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# TURNING POINT SOLUTIONS FOR THIN SHELL VIBRATIONS

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**Abstract**—Asymptotic solutions of the equations representing the free vibration of thin shells of revolution with a first-order turning point are developed. These solutions are uniform and fully approximate, namely satisfy the accuracy of the theory of thin shells.

Three categories of the generalized function are defined, $\mathscr{T}_h$ ($h = 1, 2, 3, 4$), $\mathscr{H}$ and $\mathscr{J}$, in terms of which a singular membrane solution and four bending solutions can be expressed, respectively.

In particular, the second category of the generalized function $\mathscr{H}$ is first obtained, which is a generalization of the new solution of the related equation. This new solution is found from the modification of the Laplace Transform Method.

## NOTATION

All the quantities are dimensionless except the characteristic length $R$, the elastic modulus $E$ and mass density $\rho$.

| | |
|---|---|
| $S$ | the first principal coordinate along the longitude |
| $R_1$, $R_2$ | the principal radii of curvature |
| $B$ | Lamé's coefficient, which is equivalent to the corresponding parallel radius |
| $h$ | the thickness of the shell |
| $\varepsilon$, $\mu$ | the thickness parameter (small parameter) |
| $\omega$ | frequency |
| $\Omega$ | the frequency parameter |
| $\nu$ | Poisson's ratio |
| $m$ | the number of waves along the circumference, which is restricted to not being too large, namely, $m \sim 0(1)$ |
| $u$, $v$, $w$ | the tangential, circumferential and normal displacement |
| $Z = \varphi(s)$ | Langer's variable |
| $\zeta = z/\mu$ | stretch Langer's variable |
| $\mathscr{T}_h(\zeta;p)$, $\mathscr{H}(\zeta;p)$, $\mathscr{J}(\zeta;p)$ | the first, second and third category of the generalized function, where $h = 1,2,3,4$; $p = 0, \pm 1, \pm 2, \ldots$ |
| $\{u_i, v_i, w_i\}$ ($i = 1, 2, 3, 4$) | displacement vectors corresponding to the four bending solutions |
| $\{u_5^{(0)}, v_5^{(0)}, w_5^{(0)}\}$ | the displacement vector corresponding to a singular solution of the membrane system |
| $\{p_5^{(0)}, q_5^{(0)}, r_5^{(0)}\}$ | the regular part of $\{u_5^{(0)}, v_5^{(0)}, w_5^{(0)}\}$ |
| $\{u_i^{(0)}, v_i^{(0)}, w_i^{(0)}\}$ | the displacement vector corresponding to regular solutions of the membrane system, where $i = 6$ for axisymmetric vibrations ($m = 0$) and $i = 6, 7, 8$ for asymmetric vibrations ($m \neq 0$) |
| $\{u_5, v_5, w_5\}$ | the displacement vector corresponding to a singular membrane solution whose leading terms are $\{u_5^{(0)}, v_5^{(0)}, w_5^{(0)}\}$ |
| $\{u_i, v_i, w_i\}$ | the displacement vector corresponding to the regular membrane solutions whose leading terms are $\{u_i^{(0)}, v_i^{(0)}, w_i^{(0)}\}$, where $i = 6$ for axisymmetric vibrations ($m = 0$) and $i = 6, 7, 8$ for asymmetric vibrations ($m \neq 0$) |

superscript (0) denotes zero-order or primary approximation.

## 1. INTRODUCTION

In shell vibrations, there is a turning point whose position depends on the frequency. Ross (1966) defined the turning point as any point at which the asymptotic approximation to four bending solutions is singular. Another definition was given by Gol'denveizer (1980) according to the solvability condition of the membrane system, which is obtained by putting $h \to 0$ in the original system of equations. At the turning point, the membrane system has a singularity. These two definitions are virtually equivalent to each other. The various types of vibration with turning points in solid mechanics, including shell vibrations, were discussed by Steele (1976).

The first-order turning point which occurs in shell vibrations is more complicated than that in other fields of mechanics. The complexity comes largely from the following two aspects. First, the corresponding related equation denoted by Langer's variable is of fifth order, which is much higher. In fact, the Orr–Sommerfeld equation of hydrodynamic stability corresponds to a related equation of second-order, which is the well-known Airy equation, and the related equation in toroid strength analysis is at most of third-order. However, a second more important aspect is that the membrane system is singular at the turning point. As a result, one of the two (axisymmetric vibrations) or four (asymmetric vibrations) solutions of the membrane system is singular and contains a logarithmic term at the turning point since the indicial equation arising from a Forbenius expansion at the turning point will have repeated roots. Thus, the turning point of the original system is also the branch point of the singular solution of the membrane system. In this sense, we say that the turning point in shell vibrations has a singularity at the branch point. The Orr–Sommerfeld equation has a similar nature, where the turning point is also the branch point of the solutions of the inviscous equation. However, the situation in toroid strength analysis is different. The difficulty arising from this is in finding a solution to a related equation, that can be used to uniformly describe the singular membrane solution to the high-order approximation. Here, the singular membrane solution is defined as the solution of the original system whose zero-order approximation is the singular solution of the membrane system.

The first-order turning point in shell vibrations has been investigated for many years. Ross (1966) obtained matching asymptotic solutions for the axisymmetric vibrations of shells of revolution, and discussed the possibility of finding uniform asymptotic representations. He focused his attention on the logarithmic membrane solution. However, he concluded that the logarithmic membrane solution does not appear to possess a uniform representation in any sense. In 1979, Gol'denveizer *et al.* published a monograph in Russian (Gol'denveizer *et al.*, 1979) which was really a survey of previous papers in Russian literature on the asymptotic solutions of shell vibrations, including our subject. It can be seen from this monograph that the authors tried without success to obtain a uniform representation of the singular membrane solution, although progress was made in finding that of the four bending solutions.

It is worth mentioning that a category of generalized Airy functions was introduced by Drazin and Reid (1981), to deal with the inner expansions of the singular inviscous solution of the Orr–Sommerfeld equation, whose singularity is similar to that of our singular membrane solution. However, this generalized Airy function was not recognized as a solution of the generalized Airy equation which is the generalization of the related equation of the Orr–Sommerfeld equation. This recognition is theoretically important, since only solutions of the related equation are considered to have qualitatively the same behavior as the solutions of the original equations, and can thus be used to express the latter.

In the present paper, three categories of the generalized function are defined in terms of which a singular membrane solution and four bending solutions can be expressed, respectively.

In particular, the second category of the generalized function is obtained first which is a generalization of the new solution of the related equation in shell vibrations, and can be used to uniformly expand the singular membrane solution. This new solution is found by modifying the Laplace Transform Method, which would probably provide an effective approach to finding the uniform solutions of the equations with the property that their reduced equations, obtained by putting the small parameter $\varepsilon \to 0$ in the original system of equations, are singular at the turning point of first order.

## 2. THE SYSTEM OF EQUATIONS

After the substitutions of

$$X_i = -2h\rho \frac{\partial^2 u_i}{\partial t^2} \quad (i = 1, 2, 3)$$

and

$$u_1(s, \varphi, t) = u(s)\cos m\varphi \sin \omega t$$

$$u_2(s, \varphi, t) = v(s)\sin m\varphi \sin \omega t$$

$$u_3(s, \varphi, t) = w(s)\cos m\varphi \sin \omega t$$

for the surface loads $X_i$ and the displacements $u_i$ in the equations for thin shells of revolution in terms of displacements, the system of equations representing the free vibration of thin shells of revolution may be denoted as follows:

$$(\mathbb{L} + \mu^5\mathbb{N})\mathbb{U} = -(1-\nu^2)\Omega\mathbb{U}; \tag{1}$$

where the ordinary differential operators associated with Sanders' theory of thin shells (Budiansky and Sanders, 1963) are

$$\mathbb{U} = \{U_1, U_2, U_3\} = \{u, v, w\}, \quad \mathbb{L} = [\mathbb{L}_{ij}]_{3\times 3}, \quad \mathbb{N} = [\mathbb{N}_{ij}]_{3\times 3}$$

$$\mathbb{L}_{11} = \mathbb{d}^2 + \frac{B'}{B}\mathbb{d} - \frac{\nu}{R_1 R_2} - \left(\frac{B'}{B}\right)^2 - \frac{1-\nu}{2}\left(\frac{m}{B}\right)^2$$

$$\mathbb{L}_{12} = \frac{1+\nu}{2}\frac{m}{B}\mathbb{d} - \frac{3-\nu}{2}\frac{B'}{B}\frac{m}{B}$$

$$\mathbb{L}_{13} = \left(\frac{1}{R_1} + \frac{\nu}{R_2}\right)\mathbb{d} + \frac{B'}{B}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) + \left(\frac{1}{R_1}\right)'$$

$$\mathbb{L}_{21} = -\frac{1+\nu}{2}\frac{m}{B}\mathbb{d} - \frac{3-\nu}{2}\frac{B'}{B}\frac{m}{B}$$

$$\mathbb{L}_{22} = \frac{1-\nu}{2}\mathbb{d}^2 + \frac{1-\nu}{2}\frac{B'}{B}\mathbb{d} + \frac{1-\nu}{2}\frac{1}{R_1 R_2} - \frac{1-\nu}{2}\left(\frac{B'}{B}\right)^2 - \left(\frac{m}{B}\right)^2$$

$$\mathbb{L}_{23} = -\frac{m}{B}\left(\frac{\nu}{R_1} + \frac{1}{R_2}\right)$$

$$\mathbb{L}_{31} = -\left(\frac{1}{R_1} + \frac{\nu}{R_2}\right)\mathbb{d} - \left(\frac{\nu}{R_1} + \frac{1}{R_2}\right)\frac{B'}{B}$$

$$\mathbb{L}_{32} = -\left(\frac{\nu}{R_1} + \frac{1}{R_2}\right)\frac{m}{B}$$

$$\mathbb{L}_{33} = -\left(\frac{1}{R_1} + \frac{\nu}{R_2}\right)\frac{1}{R_1} - \left(\frac{\nu}{R_1} + \frac{1}{R_2}\right)\frac{1}{R_2}$$

$$\mathbb{N}_{13} = -\frac{1}{R_1}\mathbb{d}^3 + \cdots$$

$$\mathbb{N}_{33}\mathbb{U}_3 = \frac{1}{B}\frac{\mathrm{d}}{\mathrm{d}s}\left[-B\frac{\mathrm{d}^3 w}{\mathrm{d}s^3} + \nu B\frac{\mathrm{d}}{\mathrm{d}s}\left\{\left(\frac{m}{B}\right)^2 w\right\} - \nu B\frac{\mathrm{d}}{\mathrm{d}s}\left\{\frac{B'}{B}\frac{\mathrm{d}w}{\mathrm{d}s}\right\}\right]$$
$$+ \frac{1-\nu}{B}\frac{\mathrm{d}}{\mathrm{d}s}\left[B'\left\{-\frac{\mathrm{d}^2 w}{\mathrm{d}s^2} - \left(\frac{m}{B}\right)^2 w + \frac{B'}{B}\frac{\mathrm{d}w}{\mathrm{d}s}\right\} + m^2\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{w}{B}\right)\right]$$
$$+ (1-\nu)\frac{m}{B}\left[\frac{\mathrm{d}}{\mathrm{d}s}\left\{m\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{w}{B}\right)\right\} + 2\frac{B'}{B}m\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{w}{B}\right)\right] - \left(\frac{m}{B}\right)^2\left[\left(\frac{m}{B}\right)^2 w - \frac{B'}{B}\frac{\mathrm{d}w}{\mathrm{d}s} - \nu\frac{\mathrm{d}^2 w}{\mathrm{d}s^2}\right]$$

$$\mathbb{d} = \mathrm{d}/\mathrm{d}s, \quad (\ ) = \mathrm{d}(\ )/\mathrm{d}s.$$

Moreover, $\mathbb{N}_{ij}$ $(i = 1,2,3;\ j = 1,2,3;\ i+j = 6)$ and the lower-order differential terms of $\mathbb{N}_{33}$ are neglected because they have no contribution to our subject. However, it is possible to contain the different terms for $\mathbb{N}_{ij}$ for the different thin shell theories. For example, every value of $\mathbb{N}_{ij}$, except $\mathbb{N}_{33}$, is neglected in the Gol'denveizer operators (Gol'denveizer *et al.*, 1979). The frequency and thickness parameters are

$$\Omega = \rho\omega^2 R^2/E \quad \text{and} \quad \mu^5 = \varepsilon^4 = h^2/12, \text{ respectively.} \tag{2}$$

It is clear that the evaluation of natural frequencies and their corresponding modes is, in fact, an eigenvalue problem of eqns (1) under an appropriate boundary condition.

When the frequency parameter $\Omega$ is in the frequency interval

$$\min_s \{R_2^{-2}(s)\} < \Omega < \max_s \{R_2^{-2}(s)\} \quad (s_1 \leqslant s \leqslant s_2), \tag{3}$$

eqns (1) have turning points. Actually, as mentioned by Gol'denveizer *et al.* (1979), eqns (1) can be rewritten as the following high-order equation including only the normal displacement $w$:

$$\mu^5\left(\sum_{k=0}^{n} d_k \frac{\mathrm{d}^k w}{\mathrm{d}s^k}\right) + \sum_{k=0}^{n-4} h_k(s)\frac{\mathrm{d}^k w}{\mathrm{d}s^k} = 0, \quad d_n = 1; \tag{4}$$

where $n = 6$ for axisymmetric vibration, $n = 8$ for asymmetric vibration, and the coefficient of the second-order derivative is

$$h_2(s) = h(s) = -(1-\nu^2)[\Omega - R_2^{-2}(s)].$$

Obviously, for any shells of revolution, except cylindrical and spherical shells, $h(s)$ has zero points when $\Omega$ is within the frequency interval (3). The zero points of $h(s)$ are defined as the turning points of the original equations (1), and the first-order zero points of $h(s)$ are the first-order turning points of eqns (1). It is assumed, in this paper, that only a first-order turning point exists.

For a certain frequency parameter $\Omega$ in interval (3), it is possible to find one, and only one, parallel $s = s_*$ which satisfies $\Omega = R_2^{-2}(s_*)$ and divides the middle surface of the shells along the longitude into three parts: $s_1 \leqslant s < s_*$, $|s - s_*| \ll 1$ and $s_* < s \leqslant s_2$. Here, $h(s)$ has positive, zero and negative values, respectively, and the corresponding solutions of eqns (1) have different behaviors. Hence, to find the solutions of eqns (1) which are uniformly valid in the whole interval $S$ $(s_1 \leqslant s \leqslant s_2)$ and satisfy the accuracy of the theory of thin shells is very difficult. Actually, so far the uniformly valid solutions have not been found.

## 3. THE RELATED EQUATION

It was Langer's idea that the uniformly valid solutions in the whole interval can be expressed merely in terms of nonelementary functions that have the same qualitative behavior as the solutions of the original equations. The nonelementary functions are the integrals of the so-called related equation which can be obtained from the Langer transformation using eqns (1) or eqn (4).

Introducing the Langer variables $z$ and $\zeta$

$$\zeta = \mu^{-1}z = \mu^{-1}\varphi(s) = \mu^{-1}\left[\frac{5}{4}\int_{s_*}^{s}\{h(s)\}^{1/4}\,\mathrm{d}s\right]^{4/5} \tag{5}$$

and the new dependent variable

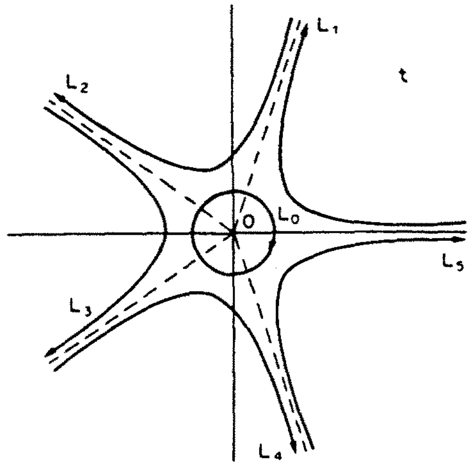


Fig. 1. Contours $L_K$ ($k = 0, 1, \ldots, 5$) in the $t$-plane.

$$y(\zeta;\mu) = \left[\frac{h(s)}{\mu\zeta}\right]^{1/8} w(s)$$

into eqn (4) and assuming that the solutions of the resulting equation can be expanded as the following asymptotic series

$$y(\zeta;\mu) = \sum_{n=0}^{\infty} \mu^n y_n(\zeta), \tag{6}$$

we obtain a series of equations as follows:

$$\begin{aligned} \mathbb{A}\mathbb{D}y_0(\zeta) &= 0 \\ \mathbb{A}\mathbb{D}y_1(\zeta) &= [\alpha(z)\mathbb{D}^5+\beta(z)]y_0(\zeta) \\ \mathbb{A}\mathbb{D}y_2(\zeta) &= [\alpha(z)\mathbb{D}^5+\beta(z)]y_1(\zeta)+\gamma(z)\mathbb{D}^4y_0(\zeta); \end{aligned} \tag{7}$$

where the differential operator

$$\mathbb{A} = \mathbb{D}^5-\zeta\mathbb{D}-2, \quad \mathbb{D} = \frac{d}{d\zeta}$$

and $\alpha(z)$, $\beta(z)$ and $\gamma(z)$ are slowly varying coefficients.

The following equation is referred to as the related equation corresponding to the original eqns (1):

$$\mathbb{A}f(\zeta) = 0, \tag{8}$$

the solutions of which are called related functions. It is interesting that operator $\mathbb{A}$ is independent of the geometrical parameters of the shells. From this it follows that all the thin shells of revolution have the same related eqn (8) and the same related functions which describe the general characteristic behavior of their free vibration.

## 4. GENERALIZED RELATED FUNCTIONS

### 4.1. *The first and third category of the generalized related function*

According to the standard Laplace approach we readily find the solutions of the related eqn (8) in integral form

$$f_K(\zeta) = \frac{1}{2\pi i}\int_{L_K} t\cdot\exp\{\zeta t - t^5/5\}\,dt \quad (k = 0, 1, 2, \ldots, 5), \tag{9}$$

where $L_K$ represents the contours in the complex $t$-plane, as shown in Fig. 1. In addition, the differentials and integrals of $f_K$ can all be expressed by

$$F_K(\zeta;p) = \frac{1}{2\pi i}\int_{L_K} t^{-p}\cdot\exp\{\zeta t - t^5/5\}\,dt \quad (k = 0, 1, 2, \ldots, 5), \tag{10}$$

where $p = 0, \pm 1, \pm 2, \ldots$ and $F_K$, respectively, represent the solutions $f_K$ of the related eqn (8) when $p = -1$, and their differentials or integrals when $p \neq -1$.

The first category of the generalized function $\mathscr{Y}_h(\zeta;p)(h = 1, 2, 3, 4)$ is defined by a combination of $F_K$ as follows:

$$\begin{aligned}
\mathscr{Y}_1 &= -iF_3\\
\mathscr{Y}_2 &= -F_2+F_4\\
\mathscr{Y}_3 &= i(-F_1+F_5)\\
\mathscr{Y}_4 &= i(F_1+F_5)+(1-i)F_0,
\end{aligned}$$

which are identically real functions for any real value of $\zeta$.

It is easily verified that $\mathscr{Y}_h$ are the solutions of the generalized related equation

$$(\mathbb{A}+p+1)\mathscr{Y}_h(\zeta;p) = 0 \tag{11}$$

and satisfy the following relations

$$\mathbb{A}\mathbb{D}\mathscr{Y}_h(\zeta;p+1) = -(p+1)\mathscr{Y}_h(\zeta;p) \tag{12}$$

$$\mathbb{D}^n\mathscr{Y}_h(\zeta;p) = \mathscr{Y}_h(\zeta;p-n) \tag{13}$$

$$\mathscr{Y}_h(\zeta;p-5)-\zeta\mathscr{Y}_h(\zeta;p-1)+(p-1)\mathscr{Y}_h(\zeta;p) = 0. \tag{14}$$

The recursion formula (14) shows that for other values of $p$, $\mathscr{Y}_h(\zeta;p)$ can be expressed as a linear combination of, for example, $\mathscr{Y}_h(\zeta;0)$, $\mathscr{Y}_h(\zeta;1), \ldots, \mathscr{Y}_h(\zeta;4)$ with polynomial coefficients.

The asymptotic expressions for $\mathscr{Y}_h(\zeta;p)$, when $\zeta \to \pm\infty$, can be obtained by the method of steepest descents as follows:

when $\zeta < 0$

$$\begin{aligned}
&\mathscr{Y}_1(\zeta;p) \to -r_p\, e^{\alpha}\{\cos(\alpha+\varphi_p)+\chi_p[\cos(\alpha+\varphi_p)+\sin(\alpha+\varphi_p)]\}\\
&\mathscr{Y}_2(\zeta;p) \to r_p\, e^{\alpha}\{\sin(\alpha+\varphi_p)-\chi_p[\cos(\alpha+\varphi_p)-\sin(\alpha+\varphi_p)]\}\\
&\qquad(\mathscr{Y}_3 \text{ and } \mathscr{Y}_4 \text{ are useless}),
\end{aligned} \tag{15$_1$}$$

and when $\zeta > 0$

$$\begin{aligned}
&\mathscr{Y}_2(\zeta;p) \to -r_p[\sin(\theta+\varphi_p)-y_p\cos(\theta+\varphi_p)]\\
&\mathscr{Y}_3(\zeta;p) \to r_p\, e^{\theta}(1+y_p)\\
&\mathscr{Y}_4(\zeta;p) \to r_p[\cos(\theta-\varphi_p)+y_p\sin(\theta-\varphi_p)]+F_0(\zeta;p)\\
&\qquad(\mathscr{Y}_1 \text{ is useless});
\end{aligned} \tag{15$_2$}$$

where

$$\alpha \sim \frac{\sqrt{2}}{2}\mu^{-5/4}|\delta|$$

$$\theta \sim \mu^{-5/4}\delta$$

$$
\begin{aligned}
\delta &= \int_{s_0}^{s} [h(s)]^{1/4}\, \mathrm{d}s \\
r_p &= \frac{1}{\sqrt{2\pi}} \mu^{(p+3/2)/4} (\tfrac{5}{4}|\delta|)^{-(p+3/2)/5} \\
\varphi_p &= \tfrac{3}{4}(p+\tfrac{3}{2})\pi \\
x_p &= \frac{\sqrt{2}}{20} \mu^{5/4} |\delta^{-1}| [\tfrac{9}{4} + p(p+4)] \\
y_p &= \tfrac{1}{10} \mu^{5/4} \delta^{-1} [\tfrac{9}{4} + p(p+4)].
\end{aligned} \tag{15$_3$}
$$

The third category of the generalized related function is defined as

$$\mathscr{I}(\zeta ; p) = F_0(\zeta ; p).$$

It is easily verified that $\mathscr{I}(\zeta ; p)$ also satisfy relations (11)–(14) with $\mathscr{Z}_h$ replaced by $\mathscr{I}$, and have the expressions:

$$\mathscr{I}(\zeta ; p) = 0 \quad (p \leqslant 0)$$

and

$$\mathscr{I}(\zeta ; p) = \sum_{n=0}^{[(p-1)/5]} \frac{(-1)^n}{5^n n!(p-5n-1)!} \zeta^{p-5n-1} \quad (p > 0); \tag{16}$$

which indicates that $\mathscr{I}(\zeta ; p)$ is a polynomial in $\zeta$ of degress $p-1$. The first few of these polynomials are

$$
\begin{aligned}
&\mathscr{I}(\zeta ; 1) = 1, &&\mathscr{I}(\zeta ; 2) = \zeta \\
&\mathscr{I}(\zeta ; 3) = \zeta^2/2!, &&\mathscr{I}(\zeta ; 4) = \zeta^3/3! \\
&\mathscr{I}(\zeta ; 5) = \zeta^4/4!, &&\mathscr{I}(\zeta ; 6) = \zeta^5/5! - 1/5 \\
&\quad \ldots
\end{aligned} \tag{17}
$$

Only four generalized related functions among $\mathscr{Z}_h$ $(h = 1, 2, 3, 4)$ and $\mathscr{I}$ are linearly independent because of the connexion formula

$$\sum_{K=0}^{5} F_K(p) = 0.$$

### 4.2. *The second category of the generalized related function*

Actually, another category of solutions of related equation (8) exists, which has the integral form (9) with the contours $L_K$ replaced by $l_K$ $(k = 1, 2, \ldots, 5)$, as shown in Fig. 2, when the standard Laplace approach is used.

However, these solutions cannot be generalized in the same manner as $f_K$ [see (9) and (10)], because the integrals (10), with contours $L_K$ replaced by $l_K$, have no significance at the origin $t = 0$ when $p > 0$.

We intend to find a new solution of the related equation (8). For this purpose, the standard Laplace approach should be modified.

Substituting the following solution

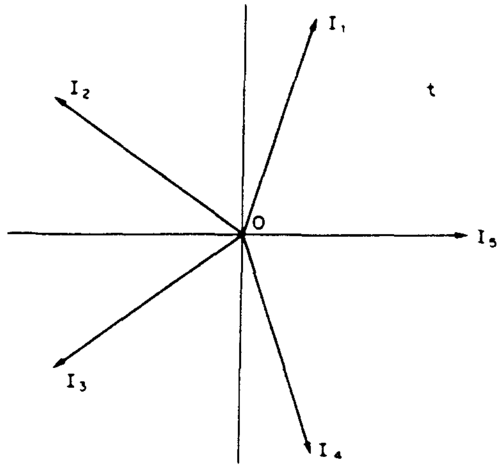


Fig. 2. The deserted contours $I_K$, on which the origin $t = 0$ lies.

$$f(\zeta) = \int_C \exp\{\zeta t\} v(t)\,\mathrm{d}t$$

into eqn (8), gives

$$-[t\cdot v\cdot \exp\{\zeta t\}]_C + \int_C \{(t^5-2)v + d(tv)/\mathrm{d}t\}\exp\{\zeta t\}\,\mathrm{d}t = 0.$$

Obviously, $f(\zeta)$ is the solution of eqn (8) when

$$[t\cdot v\cdot \exp\{\zeta t\}]_C = 0$$

and

$$(t^5-2)v + d(tv)/\mathrm{d}t = \phi(t),$$

which is not identically zero as in the standard Laplace approach, but is a single-valued, analytic function in a simply connected region surrounded by a closed contour $C+L$, where $L$ represents an auxiliary contour and satisfies

$$\int_L \phi(t)\cdot \exp\{\zeta t\}\,\mathrm{d}t = 0.$$

In our case, we choose

$$\phi(t) = t\cdot \exp\{-t^5/5\}$$

and contour $C$ to be the path in the $t$-plane which starts at an infinite point

$$\infty_K = \infty\cdot \exp\{2(k-1)\pi i/5\} \quad (k = 1, 2, \ldots, 5),$$

encircles the origin once counter-clockwise and returns to its starting point; as well as the auxiliary contour $L$ to be the circle whose center is at the origin and radius equals $R\to\infty$, as shown in Fig. 3.

In this way, a new solution of the related equation (8) is found as

$$f_K(\zeta) = \frac{1}{2\pi i}\int_{\infty_K}^{(0_+)} t\cdot\ln t\cdot\exp(\zeta t - t^5/5)\,\mathrm{d}t \quad (k = 1,2,\ldots,5) \tag{18}$$

and, in a similar manner, the second category of the generalized related function is defined as follows:

$$\mathscr{R}(\zeta;p) = \mathscr{R}_K(\zeta;p) = \frac{1}{2\pi i}\int_{\infty_K}^{(0_+)} t^{-p}\ln t\cdot\exp\{\zeta t - t^5/5\}\,\mathrm{d}t$$

$$(p = 0, \pm 1, \pm 2, \ldots;\ k = 1,2,\ldots,5);$$

where the first equals sign is valid because $\mathscr{R}_K(\zeta;p)$ is independent of $k$ (see the Appendix). This generalized function can be used to describe the singular membrane solution, which was not obtained by Gol'denveizer *et al.* (1979).

It is not difficult to verify that the second category of the generalized function satisfies the following relations, which are similar to (11)–(14),

$$\begin{aligned}(\mathbb{A}+p+1)\mathscr{R}(\zeta;p) &= \mathscr{J}(\zeta;p)\\ \mathbb{A}\mathbb{D}\mathscr{R}(\zeta;p+1) &= -(p+1)\mathscr{R}(\zeta;p)+\mathscr{J}(\zeta;p)\end{aligned} \tag{19}$$

$$\mathbb{D}^n\mathscr{R}(\zeta;p) = \mathscr{R}(\zeta;p-n) \tag{20}$$

$$\mathscr{R}(\zeta;p-5)-\zeta\mathscr{R}(\zeta;p-1)+(p-1)\mathscr{R}(\zeta;p) = \mathscr{J}(\zeta;p). \tag{21}$$

The recursion formula (21) also shows that $\mathscr{R}(\zeta;p)$ can be expressed, for different values of $p$, as a linear combination of, for example, $\mathscr{R}(\zeta;0),\ldots,\mathscr{R}(\zeta;4)$ with polynomial coefficients such as $\mathscr{L}_h$ and $\mathscr{J}$. The asymptotic expressions for $\mathscr{R}(\zeta;p)$, when $\zeta \to \pm\infty$, are

$$\begin{aligned}\mathscr{R}(\zeta;-1) &\sim \mu^2(\tfrac{5}{4}\delta)^{-8/5}\{1+\mu^5\cdot 144(\tfrac{4}{5})^4\delta^{-4}+\cdots\}\\ \mathscr{R}(\zeta;0) &\sim -\mu(\tfrac{5}{4}\delta)^{-4/5}\{1+\mu^5\cdot 24(\tfrac{4}{5})^4\delta^{-4}+\cdots\}\\ \mathscr{R}(\zeta;1) &\sim -\ln\zeta-\gamma+\frac{4!}{5}\zeta^{-5}+\cdots,\end{aligned}$$

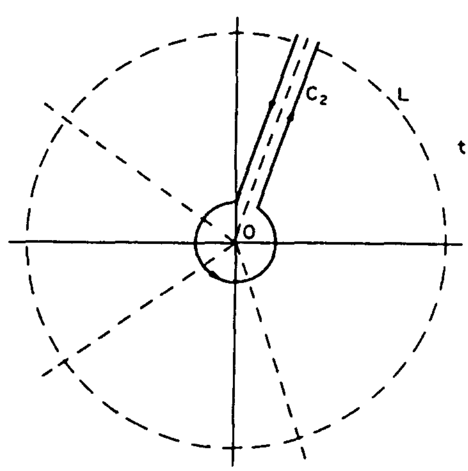


Fig. 3. Contour $C_2$ and auxiliary contour $L$ in the $t$-plane.

where $\gamma = 0.5772156649\ldots$ is Euler's constant and $\delta$ has already been given in formula (15).

Four of the first and third categories of the generalized function, and the second category are independent, which constitute the five basic solutions of the related equation (8) when $p$ equals $-1$.

## 5. THE GENERAL EXPANSION OF THE SOLUTIONS OF THE ORIGINAL EQUATIONS IN TERMS OF THREE CATEGORIES OF THE GENERALIZED FUNCTION

As shown in the previous sections, we found the solutions of eqns $(7)_1$, $(7)_2,\ldots$ step by step. The procedure is actually equivalent to the expansion of the solutions in terms of the generalized functions.

### 5.1. *Expansion of the singular membrane solution*

As can be seen from expression (18), $\mathscr{R}(\zeta;p)$ contains a factor, $\ln t$, which characterizes the singularity of the singular membrane solution at the turning point. Thus, the singular membrane solution can be expanded in terms of $\mathscr{R}(\zeta;p)$.

Letting $p = -1$ in formula (20), recalling formula $(16)_1$ and comparing the result with eqn $(7)_1$, we conclude that

$$y_0^{(5)} = \mathscr{R}(\zeta;0). \tag{22}$$

Substituting (22) into eqn $(7)_2$ and recalling the differential relation (20), we obtain

$$\mathbb{A}\mathbb{D}y_1^{(5)} = \alpha(z)\mathscr{R}(\zeta;-5)+\beta(z)\mathscr{R}(\zeta;0). \tag{23}$$

If $p = -5$ and 0 in eqn (19), respectively, and recalling eqn $(16)_1$, we immediately obtain the particular integrals of the inhomogeneous equation (23) as follows:

$$y_1^{(5)} = \gamma_1(z)\mathscr{R}(\zeta;-4)+\gamma_2(z)\mathscr{R}(\zeta;1).$$

If $p = 1$ in eqn (21), the above result can finally be rewritten as

$$y_1^{(5)} = \gamma_3(z)\zeta\mathscr{R}(\zeta;0)+\gamma_4(z)\mathscr{R}(\zeta;1)+\gamma_5(z)\mathscr{I}(\zeta;1). \tag{24}$$

Introducing $y_0^{(5)}$ and $y_1^{(5)}$ into eqn $(7)_3$ and solving the equation obtained in the same way as above, we also obtain

$$y_2^{(5)} = \delta_0(z)\zeta^2\mathscr{R}(\zeta;0)+\delta_1(z)\zeta\mathscr{R}(\zeta;1)+\delta_2(z)\mathscr{R}(\zeta;2)+\delta_3(z)\zeta\mathscr{I}(\zeta;1)+\delta_4(z)\mathscr{I}(\zeta;2)\cdots.$$

Inserting $y_0^{(5)}, y_1^{(5)}, y_2^{(5)},\ldots$ in expansion (6) means the singular membrane solution can be expressed as:

$$y^{(5)} = \tau_0(z)\mathscr{R}(\zeta;0)+\mu\tau_1(z)\mathscr{R}(\zeta;1)+\mu^2\tau_2(z)\mathscr{R}(\zeta;2)+\cdots+\mu\theta_1(z)\mathscr{I}(\zeta;1)$$
$$+\mu^2\theta_2(z)\mathscr{I}(\zeta;2)+\cdots. \tag{25}$$

However, as mentioned earlier, for $p \geqslant 5$, $\mathscr{R}(\zeta;p)$ and $\mathscr{I}(\zeta;p)$ can be expressed as a linear combination of $\mathscr{R}(\zeta;0),\ldots,\mathscr{R}(\zeta;4)$ and $\mathscr{I}(\zeta;1),\ldots,\mathscr{I}(\zeta;4)$, respectively. Thus, formula (25) is rewritten as

$$y^{(5)} = \sum_{p=0}^{4}\mu^p\pi_p(z;\mu)\mathscr{R}(\zeta;p)+\mu\sum_{p=0}^{4}\mu^p\chi_p(z;\mu)\mathscr{I}(\zeta;p+1),$$

where the second summation term can be simplied using expressions (5) and (17) as

$$\sum_{p=0}^{4} \mu^p \chi_p(z;\mu)\mathscr{I}(\zeta;p+1) = \phi(z;\mu),$$

which is the slowly varying analytic function representing the regular part of the singular membrane solution. Thus, the singular membrane solution can finally be expressed in terms of the second category of the generalized function as follows:

$$y^{(5)} = \mu\phi(z;\mu) + \sum_{p=0}^{4} \mu^p \pi_p(z;\mu)\mathscr{R}(\zeta;p), \tag{26}$$

where the slowly varying coefficients possess the following asymptotic expansions

$$\begin{aligned}\pi_p(z;\mu) &= \pi_p^{(0)}(z) + \mu^5 \pi_p^{(1)}(z) + \cdots \\ \phi(z;\mu) &= \phi^{(0)}(z) + \mu^5 \phi^{(1)}(z) + \cdots.\end{aligned} \tag{27}$$

5.2. *Expansion of the four bending solutions*

In the same way as in Section 5.1, the four bending solutions can finally be expressed as:

$$y^{(h)} = \sum_{p=0}^{4} \mu^p \pi_p(z;\mu)\mathscr{T}_h(\zeta;p) \quad (h = 1,2,3,4), \tag{28}$$

where the slowly varying coefficients are given in terms of the same notation, $\pi_p(z;\mu)$, as in expression (27), because both satisfy the same differential equations.

## 6. DETERMINATION OF THE SLOWLY VARYING COEFFICIENTS

Once the general expansions are known, finding the solutions of the original eqns (1) become the determination of the slowly varying coefficients in their expansions. Because of the restriction of the accuracy of the theory of thin shells, determination of the primary term in the asymptotic expansions (26), (28) and (27) is sufficient.

6.1. *The singular membrane solution*

From the general expansion (26), the singular membrane solution of eqns (1) is assumed to be

$$\begin{aligned}u_5(\zeta;\mu) &= \mu\phi_1(s) + \alpha_1\mathscr{R}(\zeta;0) + \mu\beta_1\mathscr{R}(\zeta;1) + \mu^2\gamma_1\mathscr{R}(\zeta;2) + \mu^3\delta_1\mathscr{R}(\zeta;3) + \mu^4\theta_1\mathscr{R}(\zeta;4) \\ v_5(\zeta;\mu) &= \mu\phi_2(s) + \alpha_2\mathscr{R}(\zeta;0) + \mu\beta_2\mathscr{R}(\zeta;1) + \mu^2\gamma_2\mathscr{R}(\zeta;2) + \mu^3\delta_2\mathscr{R}(\zeta;3) + \mu^4\theta_2\mathscr{R}(\zeta;4) \\ w_5(\zeta;\mu) &= \mu\phi_3(s) + \alpha_3\mathscr{R}(\zeta;0) + \mu\beta_3\mathscr{R}(\zeta;1) + \mu^2\gamma_3\mathscr{R}(\zeta;2) + \mu^3\delta_3\mathscr{R}(\zeta;3) + \mu^4\theta_3\mathscr{R}(\zeta;4)\end{aligned} \tag{29}$$

where the slowly varying coefficients $\phi_i$, $\alpha_i$, $\beta_i$, $\gamma_i$ and $\theta_i$ $(i = 1,2,3)$ are all functions of the original variable $s$.

Substituting (29) into eqns (1), equating the coefficients of $\mathscr{R}(\zeta;p)$ $(p = 0, \pm 1, \pm 2)$ and constant terms at both sides to each other, we obtain 18 ordinary differential equations and 18 slowly varying coefficients as unknown. Solving the equations for these coefficients yields

$$\begin{aligned}\alpha_1 &= \alpha_2 = 0 \\ \alpha_3(s) &= D_0[B(s)]^{-1/2}[\varphi'(s)]^{-3/2},\end{aligned} \tag{30}$$

where $D_0$ is a constant that remains to be determined later.

It is not necessary to find the other 15 slowly varying coefficients. If we re-expand the singular membrane solution in terms of $\mathscr{A}(\zeta;p)$ $(p = -3, -2, -1, 0, 1)$ instead of $\mathscr{A}(\zeta;p)$ $(p = 0, \pm 1, \pm 2)$ as before, and note that the terms containing $\mathscr{A}(\zeta;p)$ $(p = -3, -2, -1)$ are small quantities of higher order than the error in the theory of thin shells, we obtain

$$
\begin{aligned}
u_5 &= \mu(\phi_1+\varphi\gamma_1+\tfrac{3}{4}\varphi^2\delta_1+\tfrac{11}{36}\varphi^3\theta_1)+\mu\mathscr{A}(\zeta;1)(\beta_1+\varphi\gamma_1+\tfrac{1}{2}\varphi^2\delta_1+\tfrac{1}{6}\varphi^3\theta_1)\\
v_5 &= \mu(\phi_2+\varphi\gamma_2+\tfrac{3}{4}\varphi^2\delta_2+\tfrac{11}{36}\varphi^3\theta_2)+\mu\mathscr{A}(\zeta;1)(\beta_2+\varphi\gamma_2+\tfrac{1}{2}\varphi^2\delta_2+\tfrac{1}{6}\varphi^3\theta_2)\\
w_5 &= \mu(\phi_3+\varphi\gamma_3+\tfrac{3}{4}\varphi^2\delta_3+\tfrac{11}{36}\varphi^3\theta_3)+\mu\mathscr{A}(\zeta;1)(\beta_3+\varphi\gamma_3+\tfrac{1}{2}\varphi^2\delta_3+\tfrac{1}{6}\varphi^3\theta_3)+\alpha_3\mathscr{A}(\zeta;0);
\end{aligned}
\tag{31}
$$

which indicates that the 18 slowly varying coefficients, except $\alpha_1$, $\alpha_2$ and $\alpha_3$, appear only as six different combinations. Through a lengthy and skillful calculation, it is possible to verify that the combinations $\beta_i+\varphi\gamma_i+\frac{1}{2}\varphi^2\delta_i+\frac{1}{6}\varphi^3\theta_i$ $(i = 1, 2, 3)$ satisfy the membrane equations. On the other hand, as can be seen, the combinations are analytic. Thus, between the combinations and the three regular membrane solutions, the following relations exist

$$
\begin{aligned}
\beta_1+\varphi\gamma_1+\tfrac{1}{2}\varphi^2\delta_1+\tfrac{1}{6}\varphi^3\theta_1 &= E_1u_6^{(0)}+E_2u_7^{(0)}+E_3u_8^{(0)}\\
\beta_2+\varphi\gamma_2+\tfrac{1}{2}\varphi^2\delta_2+\tfrac{1}{6}\varphi^3\theta_2 &= E_1v_6^{(0)}+E_2v_7^{(0)}+E_3v_8^{(0)}\\
\beta_3+\varphi\gamma_3+\tfrac{1}{2}\varphi^2\delta_3+\tfrac{1}{6}\varphi^3\theta_3 &= E_1w_6^{(0)}+E_2w_7^{(0)}+E_3w_8^{(0)};
\end{aligned}
\tag{32}
$$

where $E_1$, $E_2$ and $E_3$ are constants that should be determined.

We introduce the notation to denote the other three combinations of coefficients as follows:

$$
\begin{aligned}
P_5 &= \phi_1+\varphi\gamma_1+\tfrac{3}{4}\varphi^2\delta_1+\tfrac{11}{36}\varphi^3\theta_1\\
Q_5 &= \phi_2+\varphi\gamma_2+\tfrac{3}{4}\varphi^2\delta_2+\tfrac{11}{36}\varphi^3\theta_2\\
R_5 &= \phi_3+\varphi\gamma_3+\tfrac{3}{4}\varphi^2\delta_3+\tfrac{11}{36}\varphi^3\theta_3.
\end{aligned}
\tag{33}
$$

It can also be verified that the three combinations satisfy the membrane equations in such a manner that

$$
\{\mathbb{L}+(1-\nu^2)\Omega\mathbb{I}\}\begin{pmatrix} P_5-(E_1u_6^{(0)}+E_2u_7^{(0)}+E_3u_8^{(0)})\ln\varphi\\ Q_5-(E_1v_6^{(0)}+E_2v_7^{(0)}+E_3v_8^{(0)})\ln\varphi\\ R_5-(E_1w_6^{(0)}+E_2w_7^{(0)}+E_3w_8^{(0)})\ln\varphi-\alpha_3\varphi^{-1}\end{pmatrix} = 0.
$$

On the other hand, due to the singularity of the column vector at $\varphi = 0$ we immediately conclude that the column vector must contain a component of the singular membrane solution $(u_5^{(0)}, v_5^{(0)}, w_5^{(0)})$, and have the general form

$$
\begin{aligned}
P_5-(E_1u_6^{(0)}+E_2u_7^{(0)}+E_3u_8^{(0)})\ln\varphi &= Fu_5^{(0)}+G_1u_6^{(0)}+G_2u_7^{(0)}+G_3u_8^{(0)}\\
Q_5-(E_1v_6^{(0)}+E_2v_7^{(0)}+E_3v_8^{(0)})\ln\varphi &= Fv_5^{(0)}+G_1v_6^{(0)}+G_2v_7^{(0)}+G_3v_8^{(0)}\\
R_5-(E_1w_6^{(0)}+E_2w_7^{(0)}+E_3w_8^{(0)})\ln\varphi-\alpha_3\varphi^{-1} &= Fw_5^{(0)}+G_1w_6^{(0)}+G_2w_7^{(0)}+G_3w_8^{(0)};
\end{aligned}
\tag{34}
$$

where $F$, $G_1$, $G_2$ and $G_3$ are the constants to be determined.

Substituting formulae (32), (33) and (34) into eqns (31), we obtain

$$
F = \mu^{-1}.
$$

Thus, eqns (34) can be rewritten as

$$u_5^{(0)} = \mu P_5 - \mu(G_1 + E_1 \ln\varphi)u_6^{(0)} - \mu(G_2 + E_2 \ln\varphi)u_7^{(0)} - \mu(G_3 + E_3 \ln\varphi)u_8^{(0)}$$
$$v_5^{(0)} = \mu Q_5 - \mu(G_1 + E_1 \ln\varphi)v_6^{(0)} - \mu(G_2 + E_2 \ln\varphi)v_7^{(0)} - \mu(G_3 + E_3 \ln\varphi)v_8^{(0)}$$
$$w_5^{(0)} = \mu R_5 - \mu(G_1 + E_1 \ln\varphi)w_6^{(0)} - \mu(G_2 + E_2 \ln\varphi)w_7^{(0)} - \mu(G_3 + E_3 \ln\varphi)w_8^{(0)} - \mu\alpha_3\varphi^{-1}. \tag{35}$$

Equations (35) should be valid at the turning point due to their uniform validity. Namely, provided we expand eqns (35) asymptotically at $\varphi = 0$, the result should coincide with the singular solution of the membrane equation in the power series at the neighbourhood of the turning point $s = s_*$; which is shown as:

$$u_5^{(0)} = p_5^{(0)}(s) + u_6^{(0)} \ln|s - s_*|$$
$$v_5^{(0)} = q_5^{(0)}(s) + v_6^{(0)} \ln|s - s_*|$$
$$w_5^{(0)} = r_5^{(0)}(s) + w_6^{(0)} \ln|s - s_*| + (s - s_*)^{-1}.$$

Finally, we obtain

$$E_1 = -\mu^{-1}, \quad E_2 = E_3 = 0$$
$$\mu P_5 = p_5^{(0)}, \quad \mu Q_5 = q_5^{(0)}, \quad \mu R_5 = r_5^{(0)}$$

and

$$D_0 = -\mu^{-1}[B(s_*)]^{1/2}[\varphi'(s_*)]^{5/2},$$

which appear in formula (30).

Thus, the singular membrane solution (31) is finally found as

$$u_5(\zeta;\mu) = p_5^{(0)}(s) - \mathscr{H}(\zeta;1)u_6^{(0)}(s)$$
$$v_5(\zeta;\mu) = q_5^{(0)}(s) - \mathscr{H}(\zeta;1)v_6^{(0)}(s)$$
$$w_5(\zeta;\mu) = r_5^{(0)}(s) - \mathscr{H}(\zeta;1)w_6^{(0)}(s) + \alpha_3(s)\mathscr{H}(\zeta;0), \tag{36}$$

where

$$\alpha_3(s) = -\mu^{-1}[\varphi'(s_*)]^{5/2}[\varphi'(s)]^{-3/2}\left[\frac{B(s)}{B(s_*)}\right]^{-1/2}.$$

### 6.2. *The four bending solutions*

According to the general expansion (24), the bending solutions can be assumed as

$$u_h(\zeta;\mu) = \alpha_1(s)\mathscr{Z}_h(\zeta;0) + \mu\beta_1(s)\mathscr{Z}_h(\zeta;1) + \cdots$$
$$v_h(\zeta;\mu) = \alpha_2(s)\mathscr{Z}_h(\zeta;0) + \mu\beta_2(s)\mathscr{Z}_h(\zeta;1) + \mu^2\gamma_2(s)\mathscr{Z}_h(\zeta;2) + \cdots$$
$$w_h(\zeta;\mu) = \alpha_3(s)\mathscr{Z}_h(\zeta;0) + \cdots \quad (h = 1, 2, 3, 4). \tag{37}$$

Only the coefficients in the terms shown should be found, due to the limitation of the accuracy of the linear theory of thin shells.

Substituting formula (37) into eqns (1), and equating the coefficients of $\mathscr{Z}_h(\zeta;p)$ at both sides to each other, gave 15 ordinary differential equations and 15 slowly varying coefficients $\alpha_1, \beta_1, \ldots$ as the unknowns, which are the same as the equations in the previous section (6.1). From these equations, it is easy to obtain

$$\alpha_1 = \alpha_2 = \beta_2 \equiv 0$$

$$\alpha_3(s) = C_0[B(s)]^{-1/2}[\varphi'(s)]^{-3/2}$$

$$\beta_1(s) = -C_0\left(\frac{1}{R_1} + \frac{\nu}{R_2}\right)[B(s)]^{-1/2}[\varphi'(s)]^{-5/2}$$

$$\gamma_2(s) = -C_0\frac{m}{B(s)}\left(\frac{1}{R_1} - \frac{2+\nu}{R_2}\right)[B(s)]^{-1/2}[\varphi'(s)]^{-7/2},$$

where $C_0$ is a constant.

By using the uniform validity of solutions (37), i.e. in both the subintervals $s_1 \leqslant s_*$ and $s_2 \geqslant s_*$ which are far away from the turning point $s_*$, these solutions should be identically equal to those which are valid only in the two subintervals, respectively. (The latter two solutions correspond to the cases of low frequency and high frequency, respectively, as mentioned by Zhang, 1988). The constant $C_0$ is determined as

$$C_0 = \mu^{-3/8}[B(s_*)]^{1/2}[\varphi'(s_*)]^{3/2}.$$

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## APPENDIX

A1. *Asymptotic representations of* $\mathcal{F}_k(\zeta; p)$

We start from (10) and find the representations of $F_k(\zeta; p)$ as $\zeta \to \pm\infty$ in advance.

It is not difficult to obtain their four saddle points in the $t$-plane as follows:

$$t_{0m} = \zeta^{1/4} e^{(m-1)\pi i/2} \quad (m = 1, 2, 3, 4), \tag{A1}$$

and the steepest descent paths as shown in Fig. A1. The integrals with the steepest descent paths as their contours of integration have the following asymptotic representations:

$$\phi_m(p) = \frac{1}{2\pi i}\int_{C_m} t^{-p}\exp\{\zeta t - \tfrac{1}{5}t^5\}\,\mathrm{d}t \sim \frac{1}{2\pi i}\sqrt{\frac{\pi}{2}}\,t_{0m}^{-p-(3/2)}\,e^{(4/5)\zeta t_{0m}}\sum_{k=0}^{\infty}\frac{C_k^{(p)}}{8^k k!\,t_{0m}^{5k}} \tag{A2}$$

$$(m = 1, 2, 3, 4;\ p = 0, \pm 1, \pm 2, \ldots).$$

where

$$C_k^{(p)} = \frac{\mathrm{d}^{2k}}{\mathrm{d}s^{2k}}\{(1+s)^{-p}(1+s+\tfrac{1}{2}s^2+\tfrac{1}{10}s^3)^{-(2k+1/2)}\}_{s=0}.$$

The first two terms of $C_k^{(p)}$ are

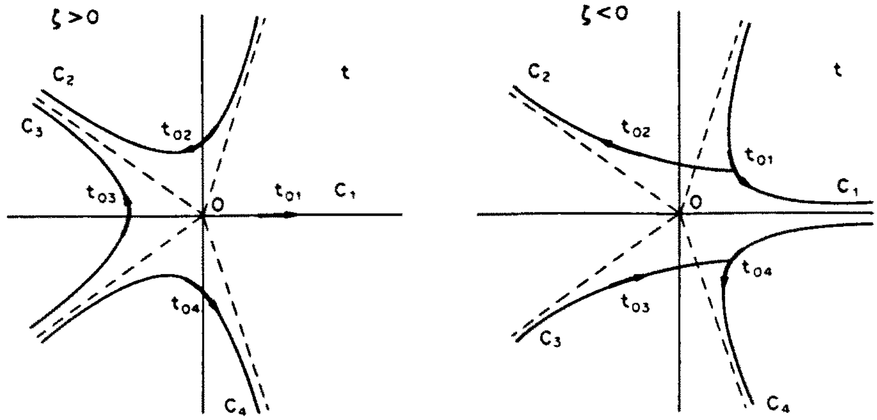


Fig. A1. Four saddle points and the steepest descent paths in the $t$-plane.

$$C_0^{(p)} = 1, \quad C_2^{(p)} = \frac{9}{4} + p(p+4) \tag{A3}$$

and the remainder are not used since the addends they appear in [see eqn (A2)] are beyond the accuracy of the theory of thin shells.

The contours of integration as given in (10) can be deformed into the steepest descent paths. Then $F_k(\zeta;p)$ as $\zeta \to +\infty$ are denoted, in terms of $\phi_m(p)$, as

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{bmatrix} \sim \begin{bmatrix} -\phi_1 - \frac{1}{2}(\phi_2 + \phi_3 - \mathcal{J}) \\ \phi_2 \\ -\phi_3 \\ \phi_4 \\ \phi_1 - \frac{1}{2}(\phi_2 + \phi_4 - \mathcal{J}) \end{bmatrix}. \tag{A4}$$

$F_k(\zeta;p)$ as $\zeta \to -\infty$ are found to be

$$\begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{bmatrix} \sim \begin{bmatrix} -\phi_1 \\ \phi_2 + \frac{1}{2}(\phi_1 + \phi_4 - \mathcal{J}) \\ -(\phi_2 + \phi_3) \\ \phi_3 + \frac{1}{2}(\phi_1 + \phi_4 - \mathcal{J}) \\ -\phi_4 \end{bmatrix}, \tag{A5}$$

where $\mathcal{J} = \mathcal{J}(\zeta;p)$ is the third generalized related function as defined before.

Clearly the values of $t_{0m}$, $\phi_m$ and $F_k$ are probably all complex although the variable $\zeta$ is real. To circumvent this disadvantage, the first generalized related function $\mathcal{T}_h(\zeta;p)$ ($h = 1, 2, 3, 4$) is defined as the following combinations of $F_k(\zeta;p)$ ($k = 1, 2, \ldots, 5$):

$$\begin{aligned} \mathcal{T}_1 &= -iF_3 \\ \mathcal{T}_2 &= -F_2 + F_4 \\ \mathcal{T}_3 &= i(-F_1 + F_5) \\ \mathcal{T}_4 &= i(F_1 + F_5) + (1-i)\mathcal{J}. \end{aligned} \tag{A6}$$

Thus, after some manipulation, the asymptotic representations of $\mathcal{T}_h(\zeta;p)$ as $\zeta \to \pm\infty$ are finally found [as given in eqns (15)].

A2. *Representations of* $\mathcal{J}(\zeta;p)$

It can be seen from the definition of $\mathcal{J}(\zeta;p)$ that $\mathcal{J}(\zeta;p) \equiv 0$ if $p \leq 0$; otherwise it is a polynomial in $\zeta$ of degree $p-1$ which, by the residue theorem, is the coefficient of $t^{p-1}$ in the expansion of $\exp\{\zeta t - \frac{1}{3}t^3\}$. The first few of these polynomials are given in eqn (17).

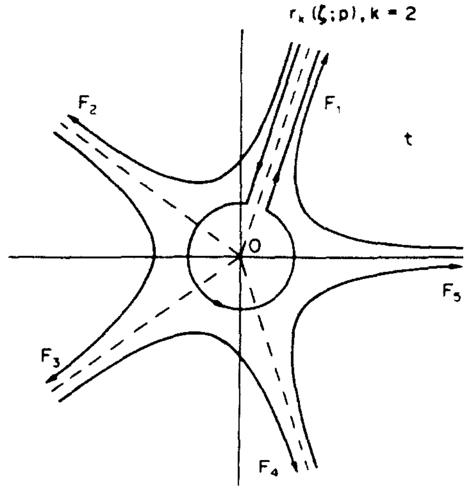


Fig. A2. Contours of $r_2(\zeta;p)$ and $F_n(\zeta;p)$ $(n = 1, 2, \ldots, 5)$.

A3. *Asymptotic representations of* $\mathcal{A}(\zeta;\mathrm{p})$

First we consider, for unrestricted (complex) values of $p$, the following integrals

$$r_k(\zeta;p) = \frac{1}{2\pi i}\int_{\infty_k}^{(0+)} t^{-p}\exp\{\zeta t - \tfrac{1}{5}t^5\}\,dt \quad (k = 1, 2, \ldots, 5). \tag{A7}$$

We observe that $r_k(\zeta;p)$ is independent of the values of $k$. In fact, for any value of $k = 1, 2, \ldots, 5$, the connexion formula

$$r_k(\zeta;p) = \sum_{n=1}^{5} F_n(\zeta;p) \quad (k = 1, 2, \ldots, 5) \text{ exists.}$$

The pattern for $k = 2$ is sketched in Fig. A2. Thus, to emphasize this independence, subscript $k$ will be ommitted.

Using a formula given by Erdélyi *et al.* (1953, p. 14), we obtain

$$r(\zeta;p) \sim \sum_{n=0}^{\infty} \frac{(-1)^n}{5^n n!\,\Gamma(p-5n)}\zeta^{p-5n-1} \quad (p \in \mathbb{C}). \tag{A8}$$

If $P$ is an integer, it is interesting to note that

$$r(\zeta;p) \equiv \mathcal{G}(\zeta;p) \quad (p = 0, \pm 1, \pm 2, \ldots). \tag{A9}$$

Thus, if $p$ is an integer, we conclude that $\mathcal{G}(\zeta;p) \equiv 0$ for all $p \leqslant 0$; otherwise the series in (A8) terminates, and

$$\mathcal{G}(\zeta;p) = \sum_{n=0}^{[(p-1)/5]} \frac{(-1)^n}{5^n n!(p-5n-1)!}\zeta^{p-5n-1} \quad (p = 1, 2, \ldots),$$

which is just formula (16).

Differentiation of (A7) with respect to $p$ $(p \in \mathbb{C})$ gives the second generalized related function $\mathcal{A}(\zeta;p) \equiv \mathcal{A}_k(\zeta;p)$, as defined before. Obversely, $\mathcal{A}(\zeta;p)$ is also independent of $k$. Thus, by differentiating both sides of (A8) and letting $p$ take an integer value again, the asymptotic representations of $\mathcal{A}(\zeta;p)$ can be found, when $\zeta \to \pm\infty$, as follows:

$$\mathcal{A}(\zeta;p) \sim \sum_{n=0}^{\infty} \frac{(-1)^{p+1}(5n-p)!}{5^n n!}\zeta^{p-5n-1} \quad (p = 0, -1, -2, \ldots)$$

and

$$\mathcal{A}(\zeta;p) \sim -(\ln\zeta + \gamma)\mathcal{G}(\zeta;p) + \sum_{n=0}^{[(p-1)/5]} \frac{(-1)^n(p-5n-1)}{5^n n!(p-5n-1)!}\sum_{k=0}^{\infty} \frac{\zeta^{p-5n+1}}{(p-5n+k)(k+1)}$$
$$+ \sum_{n=[(p-1)/5]+1}^{\infty} \frac{(-1)^{p-1}(5n-p)!}{5^n n!}\zeta^{p-5n-1} \quad (p = 1, 2, \ldots);$$

where $\gamma = 0.5772156649\ldots$ is Euler's constant.

Only $\mathcal{A}(\zeta;-1)$, $\mathcal{A}(\zeta;0)$ and $\mathcal{A}(\zeta;1)$ are useful in our subject, and their asymptotic representations have been given in Section 3.